CHARACTERIZATIONS OF BISELECTIVE OPERATIONS

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ABSTRACT. Let X be a nonempty set and let $i,j \in \{1,2,3,4\}$. We say that a binary operation $F: X^2 \to X$ is (i,j)-selective if

$$F(F(x_1,x_2),F(x_3,x_4)) = F(x_i,x_j),$$

for all $x_1,x_2,x_3,x_4 \in X$. In this paper we provide characterizations of the class of (i,j)-selective operations. We also investigate some subclasses by adding algebraic properties such as associativity or bisymmetry.

1. Introduction

Let X be a nonempty set and let $i,j \in \{1,2,3,4\}$. We say that an operation $F\colon X^2 \to X$ is (i,j)-*selective* if

$$F(F(x_1,x_2),F(x_3,x_4)) = F(x_i,x_j),$$

for all $x_1,x_2,x_3,x_4 \in X$. Also, we say that an operation $F\colon X^2 \to X$ is *biselective* if there exist $i,j \in \{1,2,3,4\}$ such that F is (i,j)-selective. Among these operations, those which are $(1,3)$-selective are of particular interest as they are *transitive*, that is, satisfy the functional equation

$$F(F(x,z),F(y,z)) = F(x,y),$$

for all $x,y,z \in X$ (see, e.g., [1,4] and the references therein). Also, we easily see that $(1,4)$-selective operations are *bisymmetric*, that is, satisfy the functional equation

$$F(F(x,y),F(u,v)) = F(F(x,u),F(y,v)),$$

for all $x,y,u,v \in X$ (see, e.g., [1]).

In this paper we investigate the class of (i,j)-selective operations for every $i,j \in \{1,2,3,4\}$. In particular, we characterize these operations with and without additional properties such as associativity or bisymmetry.

The paper is organized as follows. After presenting the main definitions, we show some basic results about (i,j)-selective operations in Section 2. In particular, we prove that (i,j)-selective operations with $j < i$ are constant (see Proposition 2.4) as well as (2,3)-selective operations (see Proposotion 2.12). We also show that characterizing the (i,j)-selective operations is equivalent to characterizing the $(5-j,5-i)$-selective operations (see Lemma 2.8). In Section 3 we characterize the $(1,3)$-selective operations (see Theorem 3.1). In Section 4 we characterize the $(1,4)$-selective operations (see Theorem 4.11) and in Section 5 we describe the $(1,2)$-selective operations in conjunction with additional properties such as

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associativity. Finally, in Section 6 we summarize the main results and present some open questions and directions of further investigations.

2. Preliminaries

In this section we introduce some basic definitions and present some preliminary results.

Definition 2.1. An operation $F: X^2 \to X$ is said to be

- *idempotent* if $F(x,x) = x$ for all $x \in X$,
- *quasitrivial* (or *selective*) if $F(x,y) \in \{x,y\}$ for all $x, y \in X$,
- *commutative* if $F(x,y) = F(y,x)$ for all $x, y \in X$,
- *anticommutative* if $\forall x, y \in X\colon\ F(x,y) = F(y,x) \Rightarrow x = y$,
- *associative* if
$$F(x, F(y,z)) \;=\; F(F(x,y),z),$$
for all $x, y, z \in X$,

Definition 2.2. Let $F: X^2 \to X$ be an operation.

- An element $e \in X$ is said to be a *neutral element* of F if $F(e,x) = F(x,e) = x$ for all $x \in X$. It can be easily shown that such a neutral element is unique.
- An element $z \in X$ is said to be an *annihilator* of F if $F(x,z) = F(z,x) = z$ for all $x \in X$. It can be easily shown that such an annihilator is unique.
- We denote the range of F by $\mathrm{ran}(F)$. Clearly, $\mathrm{ran}(F)$ is nonempty since X is nonempty.
- An element $x \in X$ is said to be *idempotent for* F if $F(x,x) = x$. We denote the set of all idempotent elements of F by $\mathrm{id}(F)$. Clearly, $\mathrm{id}(F) \subseteq \mathrm{ran}(F)$.

Recall that a binary relation R on X is said to be

- *reflexive* if $\forall x \in X\colon\ xRx$,
- *symmetric* if $\forall x, y \in X\colon\ xRy$ implies yRx,
- *transitive* if $\forall x, y, z \in X\colon\ xRy$ and yRz implies xRz.

Recall also that an *equivalence relation on* X is a binary relation $\sim$ on X that is reflexive, symmetric, and transitive. For all $u \in X$, we use the notation $[u]_\sim$ to denote the equivalence class of u, that is, $[u]_\sim = \{x \in X : x \sim u\}$.

Given $F: X^2 \to X$ we define the equivalence relation $\sim_F$ on X by
$$x \sim_F y \;\Leftrightarrow\; F(x,x) \;=\; F(y,y) \qquad x, y \in X.$$

Fact 2.3. *If $F: X^2 \to X$ is an (i,j)-selective operation, then* $\mathrm{id}(F) \cap [x]_{\sim_F} = \{F(x,x)\}$ *for all $x \in X$.*

Proposition 2.4. *An operation $F: X^2 \to X$ is an (i,j)-selective operation with $j < i$ if and only if F is constant.*

Proof. (Necessity) First, suppose that F is $(4,1)$-selective (for $(3,1)$-, $(4,2)$- and $(4,3)$-selective operations the proof is similar).

By Fact 2.3, we have $F(x,x) \in \mathrm{id}(F)$ for all $x \in X$. If $x, y \in \mathrm{id}(F)$, then
$$F(x,y) = F(F(x,x), F(y,y)) = F(y,x),$$
by $(4,1)$-selectiveness. Applying this, we get
$$x = F(x,x) = F(F(x,y), F(y,x)) = F(F(y,x), F(x,y)) = F(y,y) = y.$$

Thus, $|\mathrm{id}(F)| = 1$ and we can assume that $\mathrm{id}(F) = \{u\}$. Hence, by Fact 2.3, $F(x,x) = u$ for all $x \in X$. Using $(4,1)$-selectiveness, we get

$$F(x,y) = F(F(y,y), F(x,x)) = F(u,u) = u, \quad x, y \in X.$$

Now we suppose that F is $(2,1)$-selective (the case where F is $(4,3)$-selective can be dealt with similarly). By Fact 2.3, we have $F(x,x) \in \mathrm{id}(F)$ for all $x \in X$. If $x, y \in \mathrm{id}(F)$, then

$$x = F(x,x) = F(F(x,x), F(y,y)) = F(x,y)$$

by $(2,1)$-selectiveness. Applying that $x = F(x,y)$ and $y = F(y,x)$ for all $x, y \in \mathrm{id}(F)$, we obtain

$$x = F(x,x) = F(F(x,y), F(x,y)) = F(y,x) = y.$$

Thus, $|\mathrm{id}(F)| = 1$ and we can assume that $\mathrm{id}(F) = \{u\}$. Hence, by Fact 2.3, $F(y,y) = u$ for all $y \in X$. Using also $(2,1)$-selectiveness of F we get

$$F(y,z) = F(F(z,y), F(z,y)) = u, \quad y, z \in X.$$

(Sufficiency) Obvious. □

In the following two propositions we deal with the case where F is an (i,i)-selective operation with $i \in \{1,2,3,4\}$.

Proposition 2.5. *An operation $F\colon X^2 \to X$ is $(2,2)$-selective (resp. $(3,3)$-selective) if and only if $F|_{\mathcal{R}^2(F)}$ is constant and $F(x,x) \in \mathrm{id}(F)$ for all $x \in X$.*

Proof. (Necessity) Suppose that F is $(2,2)$-selective (the case where F is $(3,3)$-selective can be dealt with similarly). By Fact 2.3, $F(z,z) \in \mathrm{id}(F)$ for all $z \in X$. If $x, y \in \mathrm{id}(F)$, then

$$F(x,y) = F(F(x,x), F(y,y)) = F(x,x) = x.$$

Using this and $(2,2)$-selectiveness we obtain

$$x = F(x,x) = F(F(x,y), F(x,y)) = F(y,y) = y.$$

Thus, we can assume that $\mathrm{id}(F) = \{x\}$ and by Fact 2.3, $F(y,y) = x$ for all $y \in X$. Now let us assume that $u, v \in \mathrm{ran}(F)$. Then there exist $a, b, c, d \in X$ such that $u = F(a,b)$ and $v = F(c,d)$. Using $(2,2)$-selectiveness we obtain

$$F(u,v) = F(F(a,b), F(c,d)) = F(b,b) = x,$$

which proves the statement.

(Sufficiency) Obvious. □

Proposition 2.6. *An operation $F\colon X^2 \to X$ is $(1,1)$-selective (resp. $(4,4)$-selective) if and only if the following conditions hold.*

(a) *$F(x,y) = F(x,x)$ (resp. $F(x,y) = F(y,y)$) for all $x, y \in \mathrm{ran}(F)$.*
(b) *$F(x,y) \in [x]_{\sim_F}$ (resp. $F(x,y) \in [y]_{\sim_F}$) for all $x, y \in X$.*

Proof. Suppose that F is $(1,1)$-selective (the case where F is $(4,4)$-selective can be dealt with similarly).

(Necessity) If $x, y \in \mathrm{ran}(F)$, then there exists $a, b, c, d \in X$ such that $F(x,y) = F(F(a,b), F(c,d)) = F(a,a)$ by $(1,1)$-selectiveness. Also, by $(1,1)$-selectiveness,

$$F(x,x) = F(F(x,y), F(x,y)) = F(F(a,a), F(a,a)) = F(a,a),$$

which gives that $F(x,y)=F(x,x)$.

Also, by $(1,1)$-selectiveness, we have $F(F(x,y),F(x,y))=F(x,x)$, which shows that $F(x,y)\in[x]_{\sim_F}$ for all $x,y\in X$.

(Sufficiency) Let $x,y,u,v\in X$. By condition (a), we obtain

$$F(F(x,y),F(u,v))=F(F(x,y),F(x,y)).$$

Also, by condition (b), we obtain $F(F(x,y),F(x,y))=F(x,x)$. □

Remark 1. In Figure 1, we illustrate the partitioning of X by $\sim_F$, where $F\colon X^2\to X$ is $(1,1)$-selective.

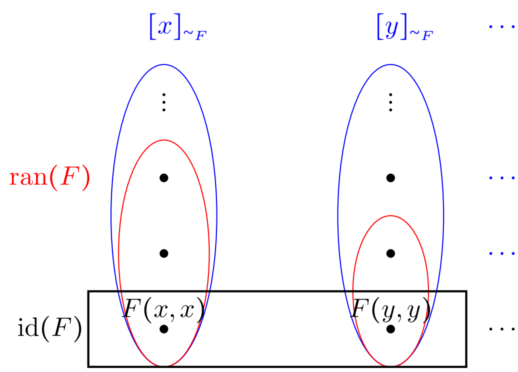

FIGURE 1.

Definition 2.7. We say that an operation $F\colon X^2\to X$ is *dual transitive* if

$$F(F(x,y),F(x,z)) \;=\; F(y,z),$$

for all $x,y,z\in X$.

The following lemma shows a strong connection between (i,j)-selective and $(5-j,5-i)$-selective operations. The proof is omitted as it is straightforward.

Lemma 2.8. *An operation $F:X^2\to X$ is (i,j)-selective (resp. transitive) if and only if the operation $G:X^2\to X$ defined by $G(x,y)=F(y,x)$ for all $x,y\in X$ is $(5-j,5-i)$-selective (resp. dual transitive).*

By Lemma 2.8, all the results for $(5-j,5-i)$-selective operations can be deduced from those for (i,j)-selective operations. Therefore we can focus only on $(1,2)$-, $(1,3)$-, $(1,4)$-, and $(2,3)$-selective operations. Now we prove some useful lemmas concerning these operations.

Recall that the *projection operations* are the binary operations $\pi_1:X^2\to X$ and $\pi_2:X^2\to X$ defined by $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for all $x,y\in X$.

The following result provides a characterization of the $(1,2)$-selective operations. Its proof is omitted as it is straightforward.

Lemma 2.9. *An operation $F\colon X^2\to X$ is $(1,2)$-selective if and only if $F|_{\mathrm{ran}(F)^2}=\pi_1|_{\mathrm{ran}(F)^2}$.*

Lemma 2.10. *Let $F: X^2 \to X$ be an operation that is $(1,4)$-selective (resp. $(1,3)$-selective, $(2,3)$-selective). For all $x, y \in X$ we have $F(x,y) = F(y,x)$ if and only if $F(x,x) = F(y,y)$. Moreover, if there exist $x, y \in X$ such that any of the previous equalities hold, then $F(x,y) = F(y,x) = F(x,x) = F(y,y)$.*

Proof. We consider the case when F is $(1,4)$-selective (the other cases can be dealt with similarly).

(Necessity) Suppose that $F(x,y) = F(y,x)$, then

$$F(x,y) = F(F(x,y), F(x,y)) = F(F(x,y), F(y,x)) = F(x,x).$$

Similarly, we have that $F(x,y) = F(y,y)$.

(Sufficiency) Now assume that $F(x,x) = F(y,y)$, then

$$F(x,x) = F(F(x,x), F(x,x)) = F(F(x,x), F(y,y)) = F(x,y).$$

Similarly, we have that $F(x,x) = F(y,x)$.

The last statement of the lemma is now immediate. □

Theorem 2.11. *Let $F: X^2 \to X$ be an (i,j)-selective operation. Then the following assertions are equivalent.*

(i) *F is commutative.*
(ii) $|\mathrm{ran}(F)| = 1$.
(iii) *F has an annihilator.*
(iv) $|\mathrm{id}(F)| = 1$.

Moreover, we have that F has a neutral element if and only if $|X| = 1$.

Proof. By Lemma 2.8 we only need to prove the result for (i,j)-selective operations where $(i,j) \in \{(1,2),(1,3),(1,4),(2,3)\}$. First, suppose that F is $(1,4)$-selective (the cases where F is $(1,3)$-selective or $(2,3)$-selective are similar).

(i) ⇒ (ii). If F is commutative, then by Lemma 2.10, $F(x,y) = F(x,x)$ for all $x, y \in X$. This implies that $|\mathrm{ran}(F)| = 1$.

(ii) ⇒ (iii). If $|\mathrm{ran}(F)| = 1$, then F has clearly an annihilator.

(iii) ⇒ (iv). Let a be the annihilator of F. We clearly have that $a \in \mathrm{id}(F)$. Also, if $x \in \mathrm{id}(F)$, then using $(1,4)$-selectiveness and the definition of an annihilator, we get $x = F(x,x) = F(F(x,a), F(a,x)) = F(a,a) = a$ which shows that $|\mathrm{id}(F)| = 1$.

(iv) ⇒ (i). If $|\mathrm{id}(F)| = 1$, then using Lemma 2.10 we get that F is commutative.

Let us now prove the last part of the statement. If $|X| = 1$, then F has clearly a neutral element. Conversely, if F has a neutral element $e \in X$, then by Lemma 2.10, $x = F(x,e) = F(e,e) = e$ for all $x \in X$ which clearly implies that $|X| = 1$.

Now, suppose that F is $(1,2)$-selective.

(i) ⇒ (ii). If $a, b \in R(F)$, then using commutativity of F and Lemma 2.9 we get $a = F(a,b) = F(b,a) = b$, which shows that $|\mathrm{ran}(F)| = 1$.

(ii) ⇒ (iii). If $|\mathrm{ran}(F)| = 1$, then F has clearly an annihilator.

(iii) ⇒ (iv). Let a be the annihilator of F. We clearly have that $a \in \mathrm{id}(F)$. Also, if $x \in \mathrm{id}(F)$, then using $(1,2)$-selectiveness and the definition of an annihilator, we get $x = F(x,x) = F(F(x,x), F(a,a)) = F(x,a) = a$, which shows that $|\mathrm{id}(F)| = 1$.

(iv) ⇒ (i). If $\mathrm{id}(F) = \{c\}$, then using $(1,2)$-selectiveness we get

$$F(x,y) = F(F(x,y), F(x,y)) = c = F(F(y,x), F(y,x)) = F(y,x),$$

for all $x, y \in X$.

Let us now prove the last part of the statement. If $|X| = 1$, then F has clearly a neutral element. Conversely, if F has a neutral element $e \in X$, then for all $x \in X$ we have $F(x,e) = x \in \operatorname{ran}(F)$. On the other hand, by Lemma 2.9, $F(e,x) = e$ for all $x \in \operatorname{ran}(F)$ which implies that $|X| = 1$. □

As an important consequence of Theorem 2.11, we provide the characterization of $(2,3)$-selective operations.

Proposition 2.12. *An operation $F: X^2 \to X$ is $(2,3)$-selective if and only if $|\operatorname{ran}(F)| = 1$.*

Proof. (Necessity) We first show that $F(x,y) = F(y,x)$ for all $x,y \in X$. Using four times $(2,3)$-selectiveness we obtain

$$\begin{aligned} F(x,y) &= F(F(x,x),F(y,y)) \\ &= F(F(F(y,x),F(x,y)),F(F(x,y),F(y,x))) \\ &= F(F(x,y),F(x,y)) = F(y,x), \end{aligned}$$

which proves the commutativity of F. Now, it follows from Theorem 2.11 that $|\operatorname{ran}(F)| = 1$.

(Sufficiency) Obvious. □

3. $(1,3)$-Selectiveness

In the following result we provide a characterization of $(1,3)$-selective operations. In the following $\sim_F$ denotes the same equivalence relation as in Section 2.

Theorem 3.1. *Let $F : X^2 \to X$ be an operation. Then, the following assertions are equivalent.*

(i) *F is $(1,3)$-selective.*
(ii) *$F(x,y) = F(u,v) \in [x]_{\sim_F}$ for all $x,y \in X$, $u \in [x]_{\sim_F}$, and $v \in [y]_{\sim_F}$.*
(iii) *$F(F(x,y),z) = F(x,z)$ and $F(x,F(y,z)) = F(x,y)$ for all $x,y,z \in X$.*

Proof. (i) ⇒ (ii). Let $x,y \in X$, $u \in [x]_{\sim_F}$ and $v \in [y]_{\sim_F}$. Using $(1,3)$-selectiveness we get

$$F(x,y) = F(F(x,x),F(y,y)) = F(F(u,u),F(v,v)) = F(u,v).$$

Also, using $(1,3)$-selectiveness, we get $F(F(x,y),F(x,y)) = F(x,x)$, which shows that $F(x,y) = F(u,v) \in [x]_{\sim_F}$.

(ii) ⇒ (iii). Let $x,y,z \in X$. Clearly, $x \in [x]_{\sim_F}$, $z \in [z]_{\sim_F}$, and by (ii) we get $F(x,y) \in [x]_{\sim_F}$ and $F(y,z) \in [y]_{\sim_F}$. Thus, by (ii), we obtain $F(F(x,y),z) = F(x,z)$ and $F(x,F(y,z)) = F(x,y)$.

(iii) ⇒ (i). Let $x,y,u,v \in X$. By (iii), we obtain

$$F(F(x,y),F(u,v)) = F(F(x,y),u) = F(x,u),$$

which concludes the proof. □

Remark 2. In Figure 2, we illustrate the partitioning of X by $\sim_F$, where $F: X^2 \to X$ is $(1,3)$-selective.

Corollary 3.2. *If $F: X^2 \to X$ is a $(1,3)$-selective operation, then $F(x,x) = F(y,z)$ for all $x \in X$ and $y,z \in [x]_{\sim_F}$.*

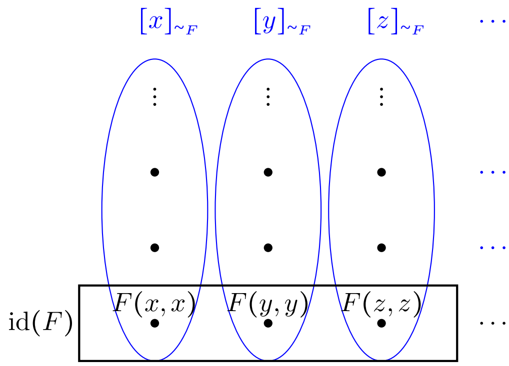

FIGURE 2.

Now if we assume that F is surjective, then we easily derive the following characterization from Theorem 3.1.

Corollary 3.3. *An operation $F\colon X^2 \to X$ is $(1,3)$-selective and surjective if and only if the following conditions hold.*

(a) *$F(x,y) = F(u,v) \in [x]_{\sim_F}$, for all $x, y \in X$, $u \in [x]_{\sim_F}$, and $v \in [y]_{\sim_F}$.*
(b) *For every $x \in X$ there exists $a, b \in X$ such that $F(F(a,a),F(b,b)) = F(a,b) = x \in [a]_{\sim_F}$.*

Remark 3. Using condition (a) of Corollary 3.3, we can reformulate condition (b) of Corollary 3.3 in the following way. If a $(1,3)$-selective operation F is surjective, then for every $x \in X$ there exists $y \in X$ such that $F(x,y) = x$.

The following result shows that associativity and bisymmetry are equivalent under $(1,3)$-selectiveness.

Proposition 3.4. *Let $F : X^2 \to X$ be an $(1,3)$-selective operation. Then the following assertions are equivalent.*

(i) *F is associative,*
(ii) *$F(x,y) = F(x,z)$ for all $x, y, z \in X$,*
(iii) *F is bisymmetric,*
(iv) *$F|_{\mathrm{ran}(F)\times X} = \pi_1|_{\mathrm{ran}(F)\times X}$.*

Proof. (i) ⇒ (ii). This follows from Theorem 3.1.

(ii) ⇒ (iii). Let $x, y, u, v \in X$. Using $(1,3)$-selectiveness and condition (ii), we obtain

$$F(F(x,y),F(u,v)) = F(x,u) = F(x,y) = F(F(x,u),F(y,v)),$$

which shows that F is bisymmetric.

(iii) ⇒ (iv). Let $x, y, z \in X$. By $(1,3)$-selectiveness, $[z]_{\sim_F} = [F(z,y)]_{\sim_F}$ for all $y \in X$. Using Theorem 3.1, bisymmetry and (1,3)-selectiveness, we obtain

$$F(F(x,y),z) = F(F(x,y),F(z,y)) = F(F(x,z),F(y,y)) = F(x,y).$$

(iv) ⇒ (i). Let $x, y, z \in X$. By Theorem 3.1 we have $F(x,y), F(x,z) \in [x]_{\sim_F}$. Thus, using Theorem 3.1 and condition (iv), we obtain

$$F(F(x,y),z) = F(x,z) = F(F(x,z),F(y,z)) = F(x,F(y,z)).$$

□

The following result provides characterizations of idempotent $(1,3)$-selective operations.

Proposition 3.5. *Let $F\colon X^2 \to X$ be an $(1,3)$-selective operation. Then the following assertions are equivalent:*

(i) *F is quasitrivial.*
(ii) *F is idempotent.*
(iii) *$|[x]_{\sim_F}| = 1$ for all $x \in X$.*
(iv) *$F = \pi_1$.*
(v) *F is anticommutative.*

Proof. (i) ⇒ (ii). Obvious.

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (iv). Let $x, y \in X$ with $x \neq y$. Since $|[x]_{\sim_F}| = |[y]_{\sim_F}| = 1$, we clearly have that x (resp. y) is the unique element of $[x]_{\sim_F}$ (resp. $[y]_{\sim_F}$). Also, by Theorem 3.1 we have $F(x,y) \in [x]_{\sim_F}$ and $F(y,x) \in [y]_{\sim_F}$ and hence $F(x,y) = x$ and $F(y,x) = y$.

(iv) ⇒ (v). Obvious.

(v) ⇒ (i). We proceed by contradiction. Suppose that there exist $x, y \in X$ such that $F(x,y) \notin \{x,y\}$. We clearly have $x \in [x]_{\sim_F}$ and by Theorem 3.1 we have $F(x,y) \in [x]_{\sim_F}$. Thus, using Corollary 3.2, we obtain $F(F(x,y),x) = F(x,x) = F(x,F(x,y))$, a contradiction. □

As an application of the structural description (see Theorem 3.1 and Figure 2) we can get the following results.

Proposition 3.6. *Let F be a $(1,3)$-selective operation on a finite X. Then*

$$|\mathrm{ran}(F)| \leq |\mathrm{id}(F)|^2.$$

Proof. By Fact 2.3, since F is $(1,3)$-selective, $F(y,y), F(z,z) \in \mathrm{id}(F)$ for all $y, z \in X$. This clearly implies that the number of ordered pairs of $\mathrm{id}(F)$ cannot be smaller than $|\mathrm{ran}(F)|$. □

The set $\mathrm{id}(F)$ can be listed as

$$\mathrm{id}(F) = \{x_1, \ldots, x_k\}$$

for some $k \geq 1$. We denote the cardinality of $[x_i]_{\sim_F}$ by l_i for all $i \in \{1, \ldots, k\}$.

Corollary 3.7. *Let F be a surjective $(1,3)$-selective operation on a finite X of cardinality $n \geq 1$ and $k = |\mathrm{id}(F)|$. Then $n \leq k^2$ and $l_i \leq k$ for all $1 \leq i \leq k$.*

Proof. This follows from Proposition 3.6 and Corollary 3.3. □

Let $s_k(n)$ denote the number of $(1,3)$-selective and surjective operations on an n-element set X with $|\mathrm{id}(F)| = k$. By Corollary 3.7, we have $s_k(n) = 0$ if $n > k^2$. Finding a closed-form expression for the number of $(1,3)$-selective or surjective $(1,3)$-selective operations seems hopeless. As an illustration of the characterization given in Theorem 3.1, we calculate $s_k(k^2)$.

Proposition 3.8. *For all integer $k \geq 1$, we have*

$$s_k(k^2) = \frac{k^2!}{((k-1)!)^{k-1}}.$$

Proof. We have $n = k^2$ and $k = |\mathrm{id}(F)|$. Hence by Corollary 3.7, $l_i = k$ for all $1 \leq i \leq k$ (see Figure 3).

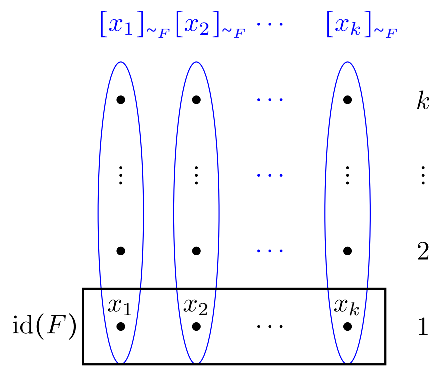

FIGURE 3.

Let X be a set with k^2 elements. First we choose the equivalence classes for $\sim_F$. All of them have k elements, thus this can be made by the multinomial coefficient $\frac{k^2!}{(k!)^k}$. Now we choose one element from each class that is also in $\mathrm{id}(F)$. This can be done in k^k different ways.

By Theorem 3.1 and surjectivity of F, for all $x_i \in \mathrm{id}(F)$ the elements of $[x_i]_{\sim_F}$ are of the form in $F(x_i, x_j)$ for some $x_j \in \mathrm{id}(F)$. Since $l_i = |\mathrm{id}(F)|$, this implies that $F(x_i, \cdot)$ is a bijection between $\mathrm{id}(F)$ and $[x_i]_{\sim_F}$ with a fixed point $x_i \in \mathrm{id}(F)$. Thus for each $i \in \{1, \ldots, k\}$ there are $(k-1)!$ different permutations. Consequently,

$$s_k(k^2) = \frac{k^2!}{(k!)^k} \cdot k^k \cdot k(k-1)! = \frac{k^2! \cdot k}{((k-1)!)^{k-1}}.$$

□

Remark 4. We observe that the number of isomorphism type of $(1,3)$-selective and surjective operations on a k^2-element set X with $|\mathrm{id}(F)| = k$ is 1.

4. $(1,4)$-SELECTIVENESS

In this section we characterize the class of operations $F: X^2 \to X$ that are $(1,4)$-selective (see Theorem 4.11).

Clearly, any operation $F: X^2 \to X$ that is diagonal bisymmetric is bisymmetric. The following result provides a characterization and partial description of the class of $(1,4)$-selective operations.

Proposition 4.1. *Let $F: X^2 \to X$ be an operation. Then, the following assertions are equivalent.*

(i) *F is $(1,4)$-selective.*
(ii) *$F|_{R(F)^2}$ is $(1,4)$-selective and $F(x,y) = F(F(x,x), F(y,y))$ for all $x, y \in X$.*
(iii) *$F(F(x,y),z) = F(x, F(y,z)) = F(x,z)$ for all $x, y, z \in X$.*

Proof. (i) ⇒ (ii). Obvious.

(ii) $\Rightarrow$ (iii). Let $x,y,z \in X$ and let us prove that $F(F(x,y),z) = F(x,z)$ (the other case can be dealt with similarly). Using our assumptions, we get

$$\begin{aligned} F(F(x,y),z) &= F(F(F(x,y),F(x,y)),F(z,z)) \\ &= F(F(F(x,y),F(x,y)),F(F(z,z),F(z,z))) \\ &= F(F(x,y),F(z,z)) \\ &= F(F(F(x,x),F(y,y)),F(F(z,z),F(z,z))) \\ &= F(F(x,x),F(z,z)) \; = \; F(x,z), \end{aligned}$$

which concludes the proof.

(iii) $\Rightarrow$ (i). Let $x,y,u,v \in X$. By assumption, we have $F(F(x,y),F(u,v)) = F(x,F(u,v)) = F(x,v)$, which shows that F is $(1,4)$-selective. □

Corollary 4.2. *If $F\colon X^2 \to X$ is $(1,4)$-selective, then it is associative.*

As another consequence we can prove the following.

Proposition 4.3. *An operation $F\colon X^2 \to X$ is $(1,4)$-selective and quasitrivial if and only if $F = \pi_1$ or $F = \pi_2$.*

Proof. (Necessity) Let $x,y \in X, x \neq y$, then we have $F(x,y) \in \{x,y\}$. We can suppose without loss of generality that $F(x,y) = x$ (the other case can be dealt with similarly). Then, by Proposition 4.1 we have $F(F(x,z),y) = F(x,y) = x$ for all $z \in X$. Thus, by quasitriviality we have $F(x,z) = x$ for all $z \in X$.

(Sufficiency) Obvious. □

Remark 5. We observe that quasitriviality cannot be relaxed into idempotency in Theorem 4.3. Indeed, let us consider $X = \{a,b,c,d\}$ and the operation $F\colon X^2 \to X$ defined by $F(a,u) = F(c,u) = a$ and $F(b,u) = F(d,u) = d$ for all $u \in \{a,d\}$ and by $F(a,v) = F(c,v) = c$ and $F(b,v) = F(d,v) = b$ for all $v \in \{b,c\}$. It is not difficult to see that F is idempotent and $(1,4)$-selective, however it is neither π_1 nor π_2. It is also not hard to show that any idempotent and $(1,4)$-selective operation on X, that is not quasitrivial, can be given as the previous operation after a suitable permutation of the elements a,b,c,d.

Now we provide a characterization of $(1,4)$-selective operations using equivalence relations.

Given $F\colon X^2 \to X$ we define two binary relations $\sim_{F,1}$ and $\sim_{F,2}$ on X by

$$x \sim_{F,1} y \iff F(x,y) = x \qquad x,y \in X,$$

and

$$x \sim_{F,2} y \iff F(x,y) = y \qquad x,y \in X.$$

Thus, given $F\colon X^2 \to X$ we have that $F|_{\{x,y\}^2} = \pi_1|_{\{x,y\}^2}$ (resp. $F|_{\{x,y\}^2} = \pi_2|_{\{x,y\}^2}$) for all $x,y \in X$ with $x \sim_{F,1} y$ (resp. $x \sim_{F,2} y$).

Remark 6. We observe that for all $F\colon X^2 \to X$ the binary relation $\sim_{F,2}$ on X was already introduced in [3, Definition 2.4] and was called the *trace* of F.

Lemma 4.4. *Let $F\colon X^2 \to X$ be a $(1,4)$-selective operation. Then, the following assertions hold.*

(a) $\mathrm{ran}(F) = \mathrm{id}(F)$.
(b) $\sim_{F,1}$ *and* $\sim_{F,2}$ *are transitive binary relations on* X.

Proof. (a). If $x \in \mathrm{id}(F)$, then we clearly have that $x \in \mathrm{ran}(F)$. Conversely, if $x \in \mathrm{ran}(F)$, then there exist $a, b \in X$ such that $x = F(a,b)$. Thus, using $(1,4)$-selectiveness, we get $F(x,x) = F(F(a,b), F(a,b)) = F(a,b) = x$, which concludes the proof.

(b). We only prove that $\sim_{F,1}$ is transitive (the other case can be dealt with similarly). Let $x, y, z \in X$ such that $x \sim_{F,1} y$ and $y \sim_{F,1} z$, that is, $F(x,y) = x$ and $F(y,z) = y$. Using diagonal bisymmetry, we get

$$F(x,z) \;=\; F(F(x,y), F(y,z)) \;=\; F(x,y) \;=\; x,$$

that is, $x \sim_{F,1} z$. □

Remark 7. We observe that Lemma 4.4(ii) is still valid if the operation F is only associative.

Proposition 4.5. *Let $F\colon X^2 \to X$ be $(1,4)$-selective. Then, the following assertions are equivalent.*

(i) *F is anticommutative.*
(ii) *F is surjective.*
(iii) *F is idempotent.*
(iv) *$\sim_{F,1}$ is an equivalence relation on X.*
(v) *$\sim_{F,2}$ is an equivalence relation on X.*

Proof. (i) ⇒ (ii). Suppose to the contrary that F is not surjective and hence not idempotent. Thus, there exists $x \in X$ such that $F(x,x) \neq x$. Using Proposition 4.1, we obtain $F(F(x,x),x) = F(x,x) = F(x, F(x,x))$, a contradiction.

(ii) ⇒ (iii). This follows from Lemma 4.4(a).

(iii) ⇒ (iv). $\sim_{F,1}$ is clearly reflexive since F is idempotent. Also, by Lemma 4.4(b), we have that $\sim_{F,1}$ is transitive. Now, let us show that $\sim_{F,1}$ is symmetric. Let $x, y \in X$ such that $x \sim_{F,1} y$, that is, $F(x,y) = x$. Then, using idempotency and $(1,4)$-selectiveness, we get

$$F(y,x) \;=\; F(F(y,y), F(x,y)) \;=\; F(y,y) \;=\; y,$$

that is, $y \sim_{F,1} x$.

(iv) ⇒ (v). $\sim_{F,2}$ is clearly reflexive since $\sim_{F,1}$ is reflexive. Also, by Lemma 4.4(b) we have that $\sim_{F,2}$ is transitive. Now, let us show that $\sim_{F,2}$ is symmetric. Let $x, y \in X$ such that $x \sim_{F,2} y$, that is, $F(x,y) = y$. By Proposition 4.1, we have that $y \sim_{F,2} F(y,x)$ and by transitivity of $\sim_{F,2}$ we get $x \sim_{F,2} F(y,x)$, that is, $F(x, F(y,x)) = F(y,x)$. By Proposition 4.1 and symmetry of $\sim_{F,1}$, we also have that $x \sim_{F,1} F(y,x)$, that is, $F(x, F(y,x)) = x$. Hence $F(y,x) = x$, that is, $y \sim_{F,2} x$.

(v) ⇒ (i). Let $x, y \in X$ such that $F(x,y) = F(y,x)$. Since $\sim_{F,2}$ is an equivalence relation on X, it follows that F is idempotent. Thus, using $(1,4)$-selectiveness, we obtain

$$x \;=\; F(x,x) \;=\; F(F(x,y), F(y,x)) \;=\; F(F(y,x), F(x,y)) \;=\; F(y,y) \;=\; y,$$

which concludes the proof. □

The following result can be easily derived from Lemma 4.4(a) and Proposition 4.5.

Corollary 4.6. *If $F\colon X^2 \to X$ is $(1,4)$-selective, then $F|_{\mathrm{ran}(F)^2}$ satisfies any of the conditions $(i)-(v)$ of Proposition 4.5.*

The following lemma is a consequence of [6, Lemma 1] and [5, Lemma 2].

Lemma 4.7. *Let $F\colon X^2 \to X$ be associative and idempotent. Then, the following assertions are equivalent.*

(i) *$F(F(x,y),z) = F(x,z)$ for all $x,y,z \in X$.*
(ii) *$F(F(x,y),x) = x$ for all $x,y \in X$.*
(iii) *F is anticommutative.*

Proposition 4.8. *An operation $F\colon X^2 \to X$ is $(1,4)$-selective and satisfies any of the conditions $(i)-(v)$ of Proposition 4.5 if and only if it is associative, idempotent, and satisfies any of the conditions $(i)-(iii)$ of Lemma 4.7.*

Proof. (Necessity) This follows from Corollary 4.2 and Proposition 4.5.
(Sufficiency) This follows from Lemma 4.7 and Propositions 4.1 and 4.5. □

Proposition 4.9. *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) *F is $(1,4)$-selective and satisfies any of the conditions $(i)-(v)$ of Proposition 4.5.*
(ii) *$\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations on X such that $[x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}} = \{F(x,y)\}$ for all $x,y \in X$.*
(iii) *The following conditions hold.*
  (a) *$\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations on X.*
  (b) *For all $x,y,z \in X$ such that $x \in [y]_{\sim_{F,1}}$ there exists a unique $u \in [z]_{\sim_{F,1}}$ such that $x \sim_{F,2} u$.*
  (c) *$F(x,y) = F(x,z)$ for all $x,y,z \in X$ such that $y \sim_{F,1} z$.*

Proof. (i) ⇒ (ii): By Proposition 4.5 we have that $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations on X. Let $x,y \in X$ and let us show that $[x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}} = \{F(x,y)\}$. By Proposition 4.1 we have $F(x,y) \in [x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}}$. Also, if $z \in [x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}}$, then $z \sim_{F,1} \cap \sim_{F,2} F(x,y)$ which by definition implies that $z = F(x,y)$.

(ii) ⇒ (iii): Condition (a) is clearly satisfied.

Let $x,y,z \in X$ such that $x \in [y]_{\sim_{F,1}}$. By (ii), we have $[x]_{\sim_{F,2}} \cap [z]_{\sim_{F,1}} = \{F(x,z)\}$, which proves condition (b).

Let $x,y,z \in X$ such that $y \sim_{F,1} z$, that is, $[y]_{\sim_{F,1}} = [z]_{\sim_{F,1}}$. By (ii) and the previous assumption, we get

$$\{F(x,y)\} = [x]_{\sim_{F,2}} \cap [y]_{\sim_{F,1}} = [x]_{\sim_{F,2}} \cap [z]_{\sim_{F,1}} = \{F(x,z)\},$$

which proves condition (c).

(iii) ⇒ (i): Since $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations on X, it follows that F is idempotent. Let $x,y,u,v \in X$ and let us show that $F(F(x,y),F(u,v)) = F(x,v)$. We clearly have that $t \in [t]_{\sim_{F,1}}$ for all $t \in X$. By conditions (b) and (c), we have that there exists a unique $s \in [y]_{\sim_{F,1}}$ such that $F(x,y) = F(x,s) = s$, that is, $x \sim_{F,2} s$. Also, by conditions (b) and (c), we have that there exists a unique $t \in [v]_{\sim_{F,1}}$ such that $F(u,v) = F(u,t) = t$, that is, $u \sim_{F,2} t$. Moreover, by conditions (b) and (c), we have that there exists a unique $z \in [v]_{\sim_{F,1}}$ such that $F(s,t) = F(s,z) = z$, that is, $s \sim_{F,2} z$. By transitivity of $\sim_{F,2}$ we have that $x \sim_{F,2} z$ and by condition (b) we have that z is unique. Thus, we obtain $F(F(x,y),F(u,v)) = F(x,v) = F(x,z) = z$ which concludes the proof. □

Remark 8. In Proposition 4.9(iii), conditions (b) and (c) can be replaced by the following two conditions.

(b') For all $x, y, z \in X$ such that $x \in [y]_{\sim_{F,2}}$ there exists a unique $u \in [z]_{\sim_{F,2}}$ such that $x \sim_{F,1} u$.
(c') $F(y,x) = F(z,x)$ for all $x, y, z \in X$ such that $y \sim_{F,2} z$.

The following corollary is an equivalent form of Proposition 4.9.

Corollary 4.10. *An operation $F: X^2 \to X$ is $(1,4)$-selective and satisfies any of the conditions $(i)-(v)$ of Proposition 4.5 if and only if the following conditions hold.*

(i) *$\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations on X and for all $x, y \in X$ there exists a bijection $f: [x]_{\sim_{F,1}} \to [y]_{\sim_{F,1}}$ defined by*

$$f(u) = v \Leftrightarrow u \sim_{F,2} v, \qquad u \in [x]_{\sim_{F,1}}, v \in [y]_{\sim_{F,1}}.$$

(ii) *$F(x,y) = F(x,z)$ for all $x, y, z \in X$ such that $y \sim_{F,1} z$.*

According to Corollary 4.10, any $(1,4)$-selective and idempotent operation $F: X^2 \to X$ can be represented in a grid form as follows. Two elements $x, y \in X$ belong to the same column (resp. row) if and only if $x \sim_{F,1} y$ (resp. $x \sim_{F,2} y$). Conversely, any operation $F: X^2 \to X$ such that $\sim_{F,1}$ and $\sim_{F,2}$ are equivalence relations and that can be represented in such a grid form with the convention that $F(x,y) = F(x,z)$ for all $x, y, z \in X$ such that $y \sim_{F,1} z$, is $(1,4)$-selective and idempotent (see Figure 4).

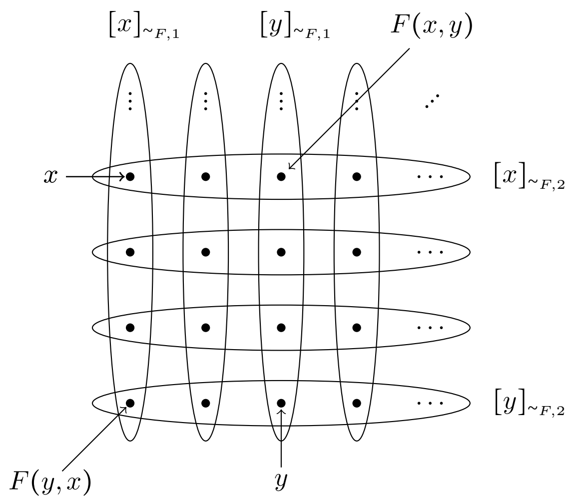

Figure 4.

The following result, which is an immediate consequence of Propositions 4.1, 4.5, 4.9 and Corollaries 4.6 and 4.10, provides a characterization of $(1,4)$-selective operations.

Theorem 4.11. *Let $F\colon X^2 \to X$ be an operation and let $F' = F|_{\mathrm{ran}(F)^2}$. Then, the following assertions are equivalent.*

(i) *F is $(1,4)$-selective*
(ii) *F' satisfies the conditions (i) – (iii) of Proposition 4.9 and $F(u,v) = F(F(u,u),F(v,v))$ for all $u,v \in X$.*
(iii) *F' satisfies the conditions (i) and (ii) of Corollary 4.10 and $F(u,v) = F(F(u,u),F(v,v))$ for all $u,v \in X$.*

For all integer $n \geq 1$, let $X_n = \{1,\ldots,n\}$ and let $\alpha(n)$ (resp. $\beta(n)$) denote the number of $(1,4)$-selective operations on X_n that are idempotent (resp. the number of isomorphism types of $(1,4)$-selective operations on X_n that are idempotent). In the following propositions we show that $\alpha(n) = A121860(n)$ and $\beta(n) = d(n) = A000005(n)$ (see [7]), where $d(n)$ denotes the number of positive integer divisors of $n \in \mathbb{N}$.

Proposition 4.12. *For all integer $n \geq 1$, we have*

$$\alpha(n) \;=\; \sum_{d|n} \frac{n!}{d!\left(\frac{n}{d}\right)!}$$

Proof. By Corollary 4.10, counting the number of $(1,4)$-selective operations on X_n that are idempotent is equivalent to counting the number of ways to partition X_n into k equivalence classes of sizes $l,\ldots,l$ and the number of bijections between two consecutive equivalence classes. Thus, we have

$$\alpha(n) \;=\; \sum_{\substack{k,l\\ kl=n}} \frac{\binom{n}{l,\ldots,l}}{k!}(l!)^{k-1} \;=\; \sum_{\substack{k,l\\ kl=n}} \frac{n!}{k!l!},$$

where the multinomial coefficient $\binom{n}{l,\ldots,l}$ provides the number of ways to put the elements $1,\ldots,n$ into k classes of sizes $l,\ldots,l$ and $l!$ is the number of bijections between two such classes. □

Proposition 4.13. *For all integer $n \geq 1$, we have $\beta(n) = d(n)$.*

Proof. By Corollary 4.10, counting the number of isomorphism types of $(1,4)$-selective operations on X_n that are idempotent is equivalent to counting the number of ways to arrange the elements of X_n in an unlabeled grid form, where two elements $x,y \in X_n$ belong to the same column (resp. row) if and only if $x \sim_{F,1} y$ (resp. $x \sim_{F,2} y$). Thus, $\beta(n)$ provides the number of ways to write n into a product of two elements $k,l \in \{1,\ldots,n\}$. This is in turn the number of divisors of n. □

Corollary 4.14. *$\alpha(n) = 2$ (resp. $\beta(n) = 2$) if and only if n is prime.*

Corollary 4.15. *Let $F\colon X_n^2 \to X_n$ be $(1,4)$-selective and idempotent. If n is prime, then $F = \pi_1$ or $F = \pi_2$.*

Remark 9. From Corollary 4.15, it follows that the example of $(1,4)$-selective and idempotent operation described in Remark 5 is the smallest example that is neither π_1 nor π_2.

5. $(1,2)$-Selectiveness

In Lemma 2.9 we already gave a characterization of $(1,2)$-selective operations. As a corollary we get the following result if F is surjective.

Corollary 5.1. *Let $F : X^2 \to X$ be an operation that is $(1,2)$-selective. Then the following assertions are equivalent.*

(i) $F = \pi_1$,
(ii) F *is quasitrivial,*
(iii) F *is idempotent,*
(iv) F *is surjective.*

Proof. (i) ⇒ (ii) ⇒ (iii) ⇒ (iv): Obvious.
(iv) ⇒ (i): This follows from Lemma 2.9. □

Now we characterize those operations that are bisymmetric.

Proposition 5.2. *A bisymmetric operation $F : X^2 \to X$ is $(1,2)$-selective if and only if the following two conditions hold.*

(i) $F(x,y) = F(x,z)$ *for all* $x,y,z \in X$,
(ii) $F|_{\mathrm{ran}(F)\times X} = \pi_1|_{\mathrm{ran}(F)\times X}$.

Proof. (Necessity) Let $x,y,z \in X$. Using bisymmetry and $(1,2)$-selectiveness, we get

$$F(x,y) = F(F(x,y),F(z,z)) = F(F(x,z),F(y,z)) = F(x,z),$$

which proves (i).

Let $x \in \mathrm{ran}(F)$ and $y \in X$. Since $x \in \mathrm{ran}(F)$, there exist $x_1,x_2 \in X$ such that $F(x_1,x_2) = x$. Hence, using (i) and $(1,2)$-selectiveness, we get

$$F(x,y) = F(x,x) = F(F(x_1,x_2),F(x_1,x_2)) = F(x_1,x_2) = x,$$

which proves (ii).

(Sufficiency) Condition (ii) clearly implies that F is $(1,2)$-selective.

Let $x,y,u,v \in X$. Using condition (i) and $(1,2)$-selectiveness, we get

$$F(F(x,y),F(u,v)) = F(x,y) = F(x,u) = F(F(x,u),F(y,v)),$$

which shows that F is bisymmetric. □

In Figure 5 we illustrate a $(1,2)$-selective and bisymmetric operation on X. The vertical lines express that $F(x,\cdot) = x$ for all $x \in \mathrm{ran}(F)$. The dotted lines express that the function F is constant along those lines.

In the following statement we provide a characterization of those operations that are associative.

Proposition 5.3. *An associative operation $F : X^2 \to X$ is $(1,2)$-selective if and only if the following two conditions hold.*

(i) $F(x,y) = F(x,F(y,z))$ *for all* $x,y,z \in X$,
(ii) $F|_{\mathrm{ran}(F)\times X} = \pi_1|_{\mathrm{ran}(F)\times X}$.

Proof. (Necessity) Let $x \in \mathrm{ran}(F)$ and $y \in X$. By Lemma 2.9 we have $F(x,x) = x$. Thus, using associativity and $(1,2)$-selectiveness, we get

$$\begin{aligned} F(x,y) &= F(F(x,x),y) \\ &= F(F(F(x,x),x),y) = F(F(x,x),F(x,y)) = F(x,x) = x, \end{aligned}$$

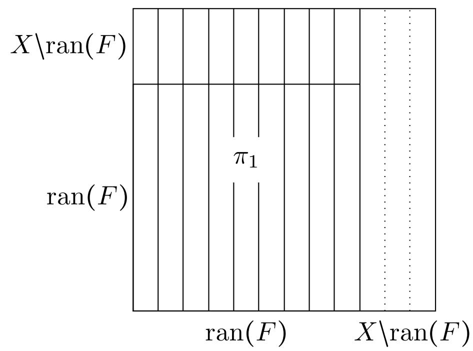

FIGURE 5.

which proves (ii).

Let $x, y, z \in X$. Since $F(x, y) \in \mathrm{ran}(F)$, using (ii) and the associativity of F, we get $F(x, y) = F(F(x, y), z) = F(x, F(y, z))$, which proves (i).

(Sufficiency) Condition (ii) clearly implies that F is $(1,2)$-selective.

Let $x, y, z \in X$. Applying (ii) for $F(x, y) \in \mathrm{ran}(F)$ and (i) we get $F(F(x, y), z) = F(x, y) = F(x, F(y, z))$, which shows that F is associative. □

As a consequence of Propositions 5.2 and 5.3 we get the following.

Corollary 5.4. *Any $(1,2)$-selective and bisymmetric operation is associative.*

Finally we show an example of $(1,2)$-selective and associative operation that is not bisymmetric on $X = \{a, b, c, d\}$ (see Figure 6). The value $F(x, y)$ is represented above the corresponding point (x, y) in Figure 6 for all $x, y \in X$. That is $F|_{\mathcal{R}_F^2} = \pi_1|_{\mathcal{R}_F^2}$ and $F(d, a) = F(d, b) = F(d, d) = a$ and $F(d, c) = b$. By Lemma 2.9, F is $(1,2)$-selective. It is also clear that F is not bisymmetric by Proposition 5.2. Using Proposition 5.2 it can be easily shown that F is associative.

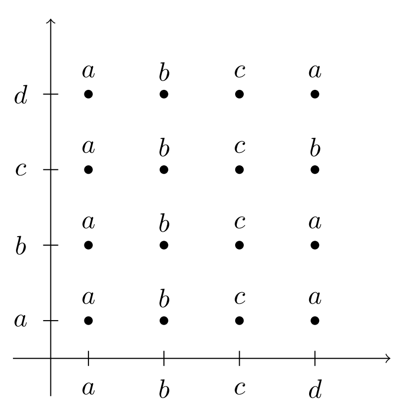

FIGURE 6.

6. Conclusion and further directions

In this article we introduced and investigated the (i,j)-selective operations. First we showed some basic properties of these operations. As a consequence we proved that any (i,j)-selective operation with $j<i$ and any $(2,3)$-selective operation is constant. We also characterized (i,i)-selective operations. We described $(1,3)$-selective operations using the equivalence relation $\sim_F$ and it turned out that it is enough to understand $\mathrm{id}(F)$ (the set of idempotent elements). We also proved that $(1,4)$-selective operations are bisymmetric and associative. We characterized $(1,4)$-selective operations using equivalence relations $\sim_{F,1}$ and $\sim_{F,2}$. Finally we described $(1,2)$-selective operations. We studied the relation of these operations with associativity, bisymmetry and other basic properties.

In view of these results some questions arise. Now, we list them below.

- Let $n\geq 3$ be an integer and let $i_1,\ldots,i_n\in\{1,\ldots,n^2\}$. We say that an operation $F\colon X^n\to X$ is $(i_1,\ldots,i_n)$*-selective*, if
$$F(F(x_1,\ldots,x_n),\ldots,F(x_{n(n-1)+1},\ldots,x_{n^2})) \;=\; F(x_{i_1},\ldots,x_{i_n}),$$
for all $x_1,\ldots,x_{n^2}\in X$.
Find characterizations of the class of $(i_1,\ldots,i_n)$-selective operations.
- Let $i,j\in\{1,2,3,4\}$. We say that the operations $F,G,H,K\colon X^2\to X$ are *generalized* (i,j)*-selective* if
$$F(G(x_1,x_2),H(x_3,x_4)) \;=\; K(x_i,x_j),\qquad x_1,x_2,x_3,x_4\in X.$$
Find characterizations of the class of generalized (i,j)-selective operations.
- Recall that an operation $F\colon X^2\to X$ is said to be *permutable* [1, 2] if it satisfies the following functional equation
$$F(F(x,y),z) \;=\; F(F(x,z),y),\qquad x,y,z\in X.$$
We observe that any $(1,2)$-selective operation that is bisymmetric is permutable. Find the conditions under which an (i,j)-selective operation is permutable.

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References

[1] J. Aczél. *Lectures in Functional Equations and Their Applications*, Dover Publications , Inc., New York, 2006.

[2] H. Bustince, M. J. Campión, F. J. Fernández, E. Induráin, and M. D.Ugarte. New trends on the permutability equation, *Aequat. Math.* **88** (2014), p. 211-232.

[3] J. C. Candeal and E. Induráin. Bivariate functional equations around associativity, *Aequat. Math.* **84** (2012), p. 137-155.

[4] Pl. Kannappan, *Functional Equations and Inequalities with Applications*, Springer, Inc., New York, 2009.

[5] N. Kimura. The structure of idempotent semigroups. I. *Pacific J. Math.*, 8:257–275, 1958.

[6] D. McLean. Idempotent semigroups. *Amer. Math. Monthly*, 61:110–113, 1954.

[7] N.J.A. Sloane (editor). The On-Line Encyclopedia of Integer Sequences. http://www.oeis.org

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